U.G. 2nd Semester Examinations 2022

MATHEMATICS (Honours)

Paper Code : MTMH DC-4

[CBCS]

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

Group - A

(4 Marks)

1. Answer any ***four*** questions : 1×4=4

   (a) If the elements a, b, ab of a group G are each of order 2, prove that $ab = ba$.

   (b) Determine whether the permutation f on the set $\{1, 2, 3, 4, 5, 6\}$ is odd or even, where $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$.

   (c) In a group $(G, \circ)$, a is an element of order 30. Find the order of a^{18}.

   (d) Are the groups $\mathbb{Z}_4$ and $\mathbb{Z}_5$ homeomorphic ? Justify your answer in brief.

   (e) Find the characteristic of the ring $(\mathbb{Z}_6, +, \bullet)$.

   (f) Determine the prime ideals of the ring $\mathbb{Z}$.

   (g) Write down the units in the ring $\mathbb{Z}_7[x]$.

Group - B

(10 Marks)

Answer any *two* questions. 5×2=10

2. Prove that a finite cyclic group $(G, *)$ of order n is cyclic if and only if there exists an element $a \in G$ such that $O(a) = n$ 5

3. Prove that a finite group G of order n is isomorphic to a subgroup of the symmetric group S_n. 5

4. Prove that every finite integral domain is a field. Hence show that if p is prime, then $(\mathbb{Z}_p, +, \bullet)$ is a field. 5

5. In a ring R, an ideal I is prime if and only if the quotient ring R/I is an integral domain. 5

Group - C

(18 Marks)

Answer any *two* questions. 9×2=18

6. (a) Let R and R' be two rings and $f : R \to R'$ be a homomorphism. Then show that the Kernel of f is an ideal of R. 5

   (b) Let G be a group and $a \in G$ be of order m.

   Prove that $O(a^k) = \dfrac{m}{\gcd(m, k)}$, where $k \in \mathbb{N}$. 4

7. (a) Prove that if every cyclic subgroup of a group G is normal, then every subgroup of G is normal. 4

   (b) Show that the set $Q(\sqrt{2}) = \{a + \sqrt{2}b : a, b \in Q\}$ forms a field under usual addition and multiplication of real numbers. 5

8. (a) Show that the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$. Show that the quotient ring $\mathbb{Z}_2[x] / \langle x^2 + x + 1 \rangle$ is a field containing 4 elements. 2+2=4

   (b) State the first isomorphism theorem. If $G_1 = (\mathbb{R}, +)$, $H = (\mathbb{Z}, +)$ and $G_2 = (\{z \in C : |z| = 1\}, \bullet)$, prove that $G_1 / H \cong G_2$. 1+4=5
