U.G. 2nd Semester Examinations 2022

MATHEMATICS (Honours)

Paper Code : MTMH DC-3

[CBCS]

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

Group - A

(4 Marks)

1. Answer any ***four*** questions : 1×4=4

(a) What is the infimum of the set $\left\{n^{(-1)^n} : n \in \mathbb{N}\right\}$?

(b) Give an example of a sequence of rational numbers having a limit which is an irrational number.

(c) Give an example of a convergent series $\sum_{n\in\mathbb{N}} a_n$ for which $\sum_{n\in\mathbb{N}} a_n^2$ is divergent.

(d) Does there always exist a fixed point of a continuous function $f:[0,1]\to[0,1]$? Justify your answer precisely.

(e) Give an example of a real-valued function which is nowhere continuous on $\mathbb{R}$.

(f) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x):=\begin{cases} x^a & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x<0. \end{cases}$$

For which values of a is f differentiable at $x = 0$?

(g) Show that Rolle's theorem is not applicable for the function $f(x)=1-(x-1)^{2/3}$, $0\leq x\leq 2$.

Group - B

(10 Marks)

Answer any *two* questions. 5×2=10

2. For every two real numbers a and b with $a < b$, prove that there exists a rational number r satisfying $a < r < b$. 5

3. Prove that a real sequence is convergent if and only if it is a Cauchy sequence. 5

4. Let f be a continuous real-valued function on a closed interval $[a, b]$. Prove that f is bounded on $[a, b]$. Also, prove that f assumes its maximum and minimum values on $[a, b]$. 2+3

5. Examine the convergence of the series $\left(\frac{1}{2}\right)^4 + \left(\frac{1.4}{2.5}\right)^4 + \left(\frac{1.4.7}{2.5.8}\right)^4 + \cdots\cdots\cdots$ 5

Group - C

(18 Marks)

Answer any *two* questions. 9×2=18

6. (a) Prove that a bounded sequence of real numbers has a convergent subsequence. 4

   (b) Show that the function f defined by $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$ but not on $(0, 1]$. 4

   (c) Evaluate the limit : $\lim\limits_{x \to \infty} e^{-x} x^2$ 1

7. (a) Check the convergence of the series $\sum\limits_{n \in \mathbb{N}} \frac{1}{n^2 - n + 1}$. 2

   (b) Let $k > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \le k|x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$. 3

   (c) If $a_{n+1} = \sqrt{k + a_n}$, where k, a_1 are positive, prove that the sequence $\{a_n\}$ is increasing or decreasing according as a_1 is less than or greater than the positive root of the equation $x^2 = x + k$ and has, in either case, this root as its limit. 4

8. (a) Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. 4

(b) Construct an example of a function $f : [0, 1] \to \mathbb{R}$ which is discontinuous at every point of $[0, 1]$ but $|f|$ is continuous on $[0, 1]$. 3

(c) Prove that an absolutely convergent series is convergent. 2